INJECTIVITY AND ACCESSIBLE CATEGORIES

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Since its creation by S. Eilenberg and S. MacLane [EM], category theory has brought a number of important concepts. Accessible categories are among them and we are going to show how they can help to treat injectivity in algebra, model theory and homotopy theory.

1 Three situation

1.1 Injective modules. Injective modules were introduced by R. Baer [B]. A left $R$-module $M$ is called injective if for each injective homomorphism $f : A \to B$ and each homomorphism $g : A \to M$ there is a homomorphism $h : B \to M$ such that $h \cdot f = g$.

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
M & & \\
\end{array} \]

The category $R$-Mod of left $R$-modules has enough injectivities, which means that for every $R$-module $A$ there is an injective homomorphism $A \to M$ with $M$ injective. This was also proved by Baer [B] using his criterion for injectivity.

**Baer's Criterion.** A left $R$-module $M$ is injective iff for every left ideal $A$ of $R$, every homomorphism $A \to R$ can be extended to a homomorphism $R \to M$. One can learn about injective modules and their use in any monograph about module theory (see, e.g., [F]).
1.2 Saturated models. Let $T$ be a first-order theory of a countable signature $\Sigma$. Let $\text{Mod}(T)$ be the category of models of the theory $T$ with elementary embeddings as morphisms. For an uncountable regular cardinal $\lambda$, a $T$-model $M$ is called $\lambda$-saturated if for each elementary embedding $f : A \to B$ with $\text{card}A, \text{card}B < \lambda$ and each elementary embedding $g : A \to M$ there is an elementary embedding $h : B \to M$ with $h \cdot f = g$.

We have not used the original definition of $\lambda$-saturated models (due to Morley and Vaught [MV]) but the characterization given in [S] 16.6. The category $\text{Mod}(T)$ has enough $\lambda$-saturated models in the sense that each $T$-model has an elementary embedding into a $\lambda$-saturated model.

1.3 Kan fibrations. The category $\text{SSet}$ of simplicial sets is defined as the functor category $\text{Set}^{\Delta^\text{op}}$ where $\Delta$ is the category of non-zero finite ordinals and order-preserving maps. The simplicial sets $\Delta^n$, $n \geq 0$ are defined as $\Delta^n = Y(n + 1)$ where $Y : \Delta \to \text{SSet}$ is the Yoneda embedding. The simplicial subsets $\Delta^n_k \subseteq \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ are obtained by excluding the identity morphism $\Delta^n \to \Delta^n$ and the morphism $\Delta^{n-1} \to \Delta^n$ given by the injective order-preserving map $n \to n + 1$ whose image does not contain $k$. A morphism $p : M \to N$ of simplicial sets is called a Kan fibration if it has the right lifting property w.r.t. each embedding $i^n_k : \Delta^n_k \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$. It means that for every commutative square

$$
\begin{array}{ccc}
\Delta^n_k & \xrightarrow{g} & M \\
\downarrow{i^n_k} & & \downarrow{p} \\
\Delta^n & \xrightarrow{\gamma} & N
\end{array}
$$

there exists a diagonal
making both triangles commutative.

If \( N = \Delta^0 \) then the unique morphism \( p : M \to \Delta^0 \) (\( \Delta^0 \) is a terminal object in \( \mathbf{SSet} \)) is a Kan fibration iff for each \( i^k_n, n \geq 0, 0 \leq k \leq n \) and for each morphism \( g : \Delta^k_n \to M \) there is a morphism \( h : \Delta^k_n \to M \) with \( h \cdot i^k_n = g \)

Such simplicial sets \( M \) are called Kan complexes. \( \mathbf{SSet} \) has enough Kan complexes in the sense that each simplicial set \( A \) has an embedding \( f : A \to B \) into a Kan complex. Moreover, this embedding \( f \) is an anodyne extension, which is defined by having the left lifting property w.r.t. each Kan fibration \( p \). It means that for every commutative square

there exists a diagonal \( h \) making both triangles commutative. Of course,
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every embedding $\Delta^n_k \to \Delta^n$ is an anodyne extension. The just explained property of having enough Kan complexes can be equivalently formulated in the way that each morphism $A \to \Delta^0$ has a factorization

$$A \xrightarrow{f} B \xrightarrow{p} \Delta^0$$

where $f$ is an anodyne extension and $p$ a Kan fibrations. More generally, every morphism $A \to N$ of simplicial sets has a factorization

$$A \xrightarrow{f} B \xrightarrow{p} N$$

where $f$ is an anodyne extension and $p$ a Kan fibration (see, e.g. [GJ]). Kan fibrations were introduced D. M. Kan [K].

2 Accessible categories

An object $K$ of a category $\mathcal{K}$ is called $\lambda$-presentable, where $\lambda$ is a regular cardinal, provided that its hom-functor $\text{hom}(K, -)$ preserves $\lambda$-directed colimits. A category $\mathcal{K}$ is called it $\lambda$-accessible provided that

1. $\mathcal{K}$ has $\lambda$-directed colimits,

2. $\mathcal{K}$ has a set $\mathcal{A}$ of $\lambda$-presentable objects such that every object is a $\lambda$-directed colimit of objects of $\mathcal{A}$.

A category is called accessible if it is $\lambda$-accessible for some regular cardinal $\lambda$. Accessible categories were introduced by C. Lair [L] and their theory was created by M. Makkai and R. Paré [MP]. We will use the monograph [AR]. The first steps towards the theory of accessible categories were made by M. Artin, A. Grothendieck and J. L. Verdier [AGV] and especially by P. Gabriel and F. Ulmer [GU].

2.1 Examples. (1) The category $R$-Mod is $\aleph_0$-accessible for every ring $R$. It has all colimits and $\aleph_0$-presentable objects are finitely presentable $R$-modules in the usual module-theoretic sense. Every $R$-module is a directed colimit of finitely presentable modules. The same argument applies to every variety of universal algebras.
(2) The category $\text{Mod}(T)$ is $\aleph_1$-accessible for every first-order theory $T$ of a countable signature. It has directed colimits (see [AR] 5.39) and $\aleph_1$-presentable objects are $T$-models having countably many elements. Every $T$-model is an $\aleph_1$-directed colimit of countable $T$-models. This can be found in [AR] 5.42 but it is an immediate consequence of the downward Löwenheim-Skolem theorem.

(3) The category $\text{SSet}$ is $\aleph_0$-accessible. It has all colimits and $\aleph_0$-presentable objects are finite colimits of simplicial sets $\Delta^n$, $n \geq 0$. Every simplicial set is a directed colimit of finite colimits of $\Delta^n$, $n \geq 0$. The same argument applies to every functor category $\text{Set}^{\mathcal{X}}$ where $\mathcal{X}$ is a small category.

(4) Let $N$ be a simplicial set and consider the comma-category $\text{SSet} \downarrow N$. Objects of this category are morphisms $p : A \to N$ of simplicial sets. Morphisms $(A, p) \to (B, q)$ are morphisms $f : A \to B$ of simplicial sets with $q \cdot f = p$.

\[
\begin{tikzcd}
A \arrow{d}[swap]{f} \arrow{r}{p} & \arrow[swap]{d}{q} N \\
B & 
\end{tikzcd}
\]

Then $\text{SSet} \downarrow N$ is an $\aleph_0$-accessible category. It has all colimits and $\aleph_0$-presentable objects are $f : A \to N$ with $A$ $\aleph_0$-presentable in $\text{SSet}$. Every object in $\text{SSet} \downarrow N$ is a directed colimit of $\aleph_0$-presentable objects (see [AR] 1.57).

Let $\mathcal{H}$ be a class of morphisms in a category $\mathcal{C}$. An object $M$ in $\mathcal{C}$ is called $\mathcal{H}$-injective if for each morphism $f : A \to B$ in $\mathcal{H}$ and each morphism $g : A \to M$ there is a morphism $h : B \to M$ such that $h \cdot f = g$.

2.2 Examples. (1) Injective $R$-modules are $\mathcal{H}$-injective objects in $R$-$\text{Mod}$ for $\mathcal{H}$ consisting of all monomorphisms.

(2) $\lambda$-saturated models are $\mathcal{H}$-injective objects in $\text{Mod}(T)$ for $\mathcal{H}$ consisting of morphisms $f : A \to B$ with $\text{card}A$, $\text{card}B < \lambda$. We recall that these objects are precisely $\lambda$-presentable objects.

(3) Kan complexes are $\mathcal{H}$-injective objects in $\text{SSet}$ for $\mathcal{H}$ consisting of anodyne extensions. In fact, we defined them as being injective w.r.t. embeddings $\Delta^n_k \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ but it immediately follows from
the definition that they are injective w.r.t. every anodyne extension.

(4) Let \( N \) be a simplicial set and consider the comma-category \( \text{SSet} \downarrow N \). Kan fibrations \( p : M \to N \) are \( \mathcal{H} \)-injective objects for \( \mathcal{H} \) consisting of morphisms \((A, a) \to (B, b)\) carried by anodyne extensions \( f : A \to B \). In fact the defining property of a Kan fibration exactly means that

\[
\begin{array}{ccc}
(A, pu) & \xrightarrow{f} & (B, v) \\
\downarrow u & & \downarrow h \\
(M, p) & & \\
\end{array}
\]

An accessible category does not need to have all colimits (see, for example, 2.1 (2)). We say that a diagram \( D : D \to K \) has a bound in a category \( K \) if there is a compatible cocone \( (DD, C_d, C)_{d \in \text{Obj}} \) in \( K \). We say that \( K \) has directed bounds if every directed diagram has a bound in \( K \) and that \( K \) has pushout bounds if every diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & & \\
\end{array}
\]

has a bound in \( K \).

2.3 Theorem. Let \( K \) be an accessible category with directed and pushout bounds and \( \mathcal{H} \) a set of morphisms in \( K \). Then every object \( K \) in \( K \) has a morphism \( K \to M \) into an \( \mathcal{H} \)-injective object \( L \).

Proof. Following [AR] 2.14 and 2.2 (3), there is a regular cardinal \( \lambda \) such that \( K \) is \( \lambda \)-accessible and every morphism in \( \mathcal{H} \) has a \( \lambda \)-presentable domain. Consider an object \( K \) in \( K \). Let \( \mathcal{X}_K \) be the set of all spans

\[
\begin{array}{ccc}
K & \xrightarrow{u} & C \\
\downarrow & & \downarrow g \\
& D & \\
\end{array}
\]

with \( g \in \mathcal{H} \). We will index these spans by ordinals \( i < \mu_K = \text{card}\mathcal{X}_K \).
We define a chain $k_{ij} : K_i \rightarrow K_j$, $i \leq j \leq \mu_K$ by the following transfinite induction:

First step: $K_0 = K$.

Isolated step: $K_{i+1}$ is given by a pushout bound

$$K_i \xrightarrow{k_{i,i+1}} K_{i+1} \leftarrow C_i \xrightarrow{\sigma_i} D_i$$

where $k_{0,i+1} = k_{i,i+1} \cdot k_{0,i}$.

Limit step: $K_i$ is a bound of the chain

$$K_0 \xrightarrow{k_{01}} K_1 \xrightarrow{k_{12}} \ldots K_j \xrightarrow{k_{j,j+1}} \ldots$$

where $j < i$ and $k_{0i} : K_0 \rightarrow K_i$ is given by this bound.

The object $K_{\mu_K}$ will be denoted by $K^*$ and the morphism $K_{0\mu_k} : K \rightarrow K^*$ by $t_K$. Following the construction, each span $(u_i, g_i) \in \mathcal{X}_K$ has a pushout bound

$$K \xrightarrow{t_k} K^*$$

We define a chain $m_{ij} : M_i \rightarrow M_j$, $i \leq j \leq \lambda$ by the following transfinite induction:

First step: $M_0 = K$.

Isolated step: $m_{i,i+1} : M_i \rightarrow M_{i+1}$ is $t_{M_i} : M_i \rightarrow M^*_i$.

Limit step: $M_i$ is a directed bound of the chain

$$M_0 \xrightarrow{m_{01}} M_1 \xrightarrow{m_{12}} \ldots M_j \xrightarrow{m_{j,j+1}} \ldots$$ (1)

for $j < i < \lambda$ and $M_{\lambda}$ is a colimit of (1) for $i = \lambda$.

We will show that $m_{0\lambda} : K \rightarrow M_{\lambda}$ is a desired morphism of $K$ into an $\mathcal{H}$-injective object. Consider a span
Since the object $C$ is $\lambda$-presentable and $M_{\lambda}$ is a directed colimit of $M_i$, $i < \lambda$, there is a factorization of $u$ through $M_i$ for some $i < \lambda$. Since the span is in the set $X_{M_i}$, it has a pushout bound. We have

\[ u = m_{i,\lambda} \cdot u' = m_{i+1,\lambda} \cdot m_{i,i+1} \cdot u' = m_{i+1,\lambda} \cdot v \cdot g. \]

Hence $u$ factorizes through $g$, which proves that $M_{\lambda}$ is $\mathcal{H}$-injective.

**2.4 Examples.** (1) The category $R$-Mod is $\aleph_0$-accessible and has all colimits. Let $\mathcal{H}$ be the set of all embeddings $A \to R$ where $A$ is a left ideal in
Following Baer’s Criterion \( \mathcal{H} \)-injective modules are precisely injective modules. Following Theorem 2.3 every \( R \)-module has a homomorphism into an injective \( R \)-module.

To prove that \( R\text{-Mod} \) has enough injectives, we have to replace the category \( R\text{-Mod} \) by the category \( R\text{-Mod}_0 \) of \( R \)-modules and injective homomorphisms taken as morphisms. Following [AR] 2.3 (6), \( R\text{-Mod}_0 \) is an accessible category. It has directed colimits (by [AR] 1.62) and pushouts because monomorphisms in \( R\text{-Mod} \) are stable under pushouts. Hence, by applying Theorem 2.3, to the category \( R\text{-Mod}_0 \), we get that \( R\text{-Mod} \) has enough injectives.

(2) Let \( T \) be a first-order theory of a countable signature and \( \lambda \) an uncountable regular cardinal. The category \( \text{Mod}(T) \) has pushout bounds (see [H], p. 288). Hence Theorem 2.3 together with Example 2.1 (2) implies that every \( T \)-model has an elementary embedding into a \( \lambda \)-saturated \( T \)-model. Of course, we take for \( \mathcal{H} \) the set of all elementary embedding \( A \to B \) with \( \text{card}A, \text{card}B < \lambda \).

(3) The category \( \text{SSet} \) is \( \aleph_0 \)-accessible and has all colimits. Let \( \mathcal{H} \) consist of embeddings \( \Delta^n_k \to \Delta^n \), \( n \geq 0, 0 \leq k \leq n \). Following Theorem 2.3, every simplicial set \( A \) has a morphism \( m : A \to M \) into a Kan complex \( M \).

Since \( \text{SSet} \) is cocomplete, we can use colimits instead of bounds in the proof of Theorem 2.3. Hence \( m \) belongs to the closure of \( \mathcal{H} \) under pushouts, compositions and colimits of chains. Every morphism of this closure belongs to \( \square(\mathcal{H}^\square) \) where the box on the right (left) means the use of the right (left) lifting property. Hence \( m \) is an anodyne extension.

More generally, by applying Theorem 2.3 to the category \( \text{SSet} \downarrow N \) (for \( \mathcal{H} \) consisting of morphism carried by embeddings \( \Delta^n_k \to \Delta^n \), \( n \geq 0, 0 \leq k \leq n \)), we get that each morphism \( A \to N \) has a factorization

\[
A \xrightarrow{f} B \xrightarrow{p} N
\]

where \( f \) is an anodyne extension and \( p \) a Kan fibration.

The last example gives the essence of essence of the small object argument already present in [GZ]. This argument is commonly used in homotopy theory (see [Ho]) but the theory of accessible categories has started to be used in homotopy theory only recently (see T. Beke [B]). Our Theorem 2.3 is a very general formulation of the small object argument. The point is that every object of an accessible category is presentable (= small), which
makes possible to stop the construction of an $\mathcal{H}$-injective object $M$ for $K$. The next example shows that it is necessary to assume that $\mathcal{H}$ is a set.

2.5 Example. Let $\text{Gr}$ be the category of groups and $\mathcal{H}$ the class of all injective homomorphisms. Every group $K$ is a subgroup of a simple group $L \neq K$ (see [Sc]). If $K$ is $\mathcal{H}$-injective, the embedding $f : K \to L$ splits, i.e., there exists $g : L \to K$ with $g \cdot f = \text{id}_K$; by applying $\mathcal{H}$-injectivity to

$$
\begin{array}{ccc}
K & \xrightarrow{id_K} & K \\
\downarrow{f} & & \downarrow{g} \\
L & \xrightarrow{g} & K
\end{array}
$$

Since $L$ is simple and $L \neq K$, the homomorphism $g$ has to be constant, i.e., $K = \{1\}$. Therefore the trivial group $\{1\}$ is the only injective ($= \mathcal{H}$-injective) group. Hence the category of groups does not have enough injectives. On the other hand, the category $\text{Gr}_0$ of groups and injective homomorphisms is accessible (following the same reasons as the category $R$-$\text{Mod}_0$) and the only obstacle to apply Theorem 2.3 is that $\mathcal{H}$ is not a set.

References


