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The Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

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ABSTRACT

The object of the paper is to study a type of canonical linear connection, called the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold.

RESUMEN

El objetivo de este artículo es estudiar un tipo de conexión lineal canónica, llamada la conexión canónica paracontacto de Zamkovoy en una variedad para-Kenmotsu.

Keywords and Phrases: Para-Kenmotsu manifold; Zamkovoy canonical paracontact connection; local ϕ-symmetry; local ϕ-Ricci symmetry; recurrent; η-Einstein manifold.

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1 Introduction

In recent years, many authors started to the study of paracontact geometry due to its unexpected relation with the most activated contact geometry. As a result of this, in 1985, S. Kaneyuki and F. L. Williams [9] introduced the notion of paracontact metric manifold as a natural counter part of the well known contact metric manifold. Since then, several authors studied these manifolds by focusing on various special cases. A systematic study of paracontact metric manifolds and their subclasses were carried out by S. Zamkovoy [25] by emphasizing similarities and differences with respect to the contact case. Further, the notion of para-Kenmotsu manifold was introduced by J. Welyczko [23] for 3-dimensional normal almost paracontact metric structures. This structure is an analogy of Kenmotsu manifold [10] in paracontact geometry. Again the similar notion called P-Kenmotsu manifold was studied by B. B. Sinha and K. L. Sai Prasad [20] and they obtained many results. At this point, we refer the papers [1, 4, 14, 15, 16, 26] and the references therein to reader for a wide and detailed overview of the results on para-Kenmotsu manifolds.

In the context of para-Kenmotsu geometry, author A. M. Blaga [2] studied certain canonical linear connections (Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections) with a special view towards ϕ-conjugation. Some properties of generalized dual connections of the above said canonical linear connections on a para-Kenmotsu manifold was also studied in [3]. As a continuation of this, we are considering one of such canonical linear connection on a para-Kenmotsu manifold. So we undertake the study of Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This connection on a paracontact manifolds was adapted and studied rigorously by S. Zamkovoy [25]. This connection plays the role of the (generalized) Tanaka-Webster connection [22] in paracontact geometry. The main feature of this connection is that, it is metrical but not symmetrical. Throughout the paper, we refer the canonical linear connection as *Zamkovoy canonical paracontact connection*.

On the other hand, the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent. In 1977, T. Takahashi [21] introduced the notion of local ϕ-symmetry on a Sasakian manifold as a weaker version of local symmetry of such a manifold. Since then, several authors studied this notion on various structures and their generalizations or extension in [6, 7, 12, 13, 17, 18, 20]. A para-Kenmotsu manifold is said to be locally ϕ-symmetric if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)U) = 0 \tag{1.1}$$

for any vector fields X, Y, U, W orthogonal to ξ on M, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Recently, U. C. De and A. Sarkar [5] introduced the notion of local ϕ-Ricci symmetry on a Sasakian manifold. Further, this notion was studied by S. Ghosh and U. C. De [8] in the context of (κ, μ)-

contact metric manifolds and obtained interesting results. A para-Kenmotsu manifold M is said to be locally ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2((\nabla_W Q)X) = 0,$$

for any vector fields X, W orthogonal to ξ on M and $S(X,W) = g(QX,W)$.

The object of the present paper is to study the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This paper is organized as follows: Section 2 is devoted to preliminaries on para-Kenmotsu manifolds. In section 3, we give a brief account of information regarding the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ on a para-Kenmotsu manifold and obtain a relationship between the Levi-Civita connection ∇ and the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. In section 4, we characterize locally ϕ-symmetric and locally concircular ϕ-symmetric para-Kenmotsu manifolds with respect to the connection $\nabla^{\mathcal{Z}}$. It is prove that the notion of local ϕ-symmetry (also, locally concircular ϕ-symmetry) with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇ are equivalent. Section 5, covers the study of locally ϕ-Ricci symmetric para-Kenmotsu manifold with respect to the connection $\nabla^{\mathcal{Z}}$ and prove that a para-Kenmotsu manifold is locally ϕ-symmetric with respect to the connection $\nabla^{\mathcal{Z}}$, then the manifold is Ricci symmetric and hence it is an Einstein manifold. A para-Kenmotsu manifold whose curvature tensor is covariant constant with respect to the connection $\nabla^{\mathcal{Z}}$ and the manifold is recurrent with respect to the connection ∇ is studied in section 6 and shown that in this situation the manifold is η-Einstein manifold. Finally, we construct an example of a 3-dimensional para-Kenmotsu manifold admitting the connection $\nabla^{\mathcal{Z}}$ to illustrate some results.

2 Preliminaries

Let M be an n-dimensional differentiable manifold, n is odd, with an almost paracontact structure (ϕ, ξ, η), that is, ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad rank(\phi) = n-1. \tag{2.2}$$

Let g be a pseudo-Riemannian metric compatible with (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y) \tag{2.3}$$

for any vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then the manifold is said to be an almost paracontact metric manifold. From (2.3) it can be easily deduce that

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X), \tag{2.4}$$

for any vector fields $X, Y \in \chi(M)$. An almost paracontact metric manifold becomes a paracontact metric manifold [25] if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g and is denoted by (M, g). If moreover,

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the pseudo-Riemannian connection of g holds, then (M, g) is called an para-Kenmotsu manifold. From (2.5), it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

Moreover, in a para-Kenmotsu manifold (M, g) of dimension n, the curvature tensor R and the Ricci tensor S satisfy [23]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

for any vector fields $X, Y, Z \in \chi(M)$.

A para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X, W) = ag(X, W) + b\eta(X)\eta(W),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M reduces to an Einstein manifold with some constant a.

3 Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

In the following, we consider a connection $\nabla^{\mathcal{Z}}$ on an almost paracontact metric manifold using the Levi-Civita connection ∇ of the structure [25]:

$$\nabla^{\mathcal{Z}}_X Y = \nabla_X Y + (\nabla_X \eta)Y.\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{3.1}$$

If we use (2.6) and (2.7) in (3.1), we obtain

$$\nabla^{\mathcal{Z}}_X Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y, \tag{3.2}$$

for any vector fields $X, Y \in \chi(M)$. We call the connection $\nabla^{\mathcal{Z}}$ defined by (3.2) on a para-Kenmotsu manifold, the *Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold*.
The expression for the curvature tensor $R_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ is defined by

$$R_{\nabla^{\mathcal{Z}}}(X,Y)U = \nabla^{\mathcal{Z}}_X \nabla^{\mathcal{Z}}_Y U - \nabla^{\mathcal{Z}}_Y \nabla^{\mathcal{Z}}_X U - \nabla^{\mathcal{Z}}_{[X,Y]} U.$$

Then, in a para-Kenmotsu manifold, we have

$$R_{\nabla^{\mathcal{Z}}}(X,Y)U = R(X,Y)U + g(Y,U)X - g(X,U)Y, \tag{3.3}$$

where $R(X,Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U$, is the curvature tensor of M with respect to the connection ∇. The expression (3.3) is treated as the curvature tensor of a para-Kenmotsu manifold with respect to the connection $\nabla^{\mathcal{Z}}$.

Proposition 3.1. *A para-Kenmotsu manifold is Ricci-flat with respect to the Zamkovoy canonical paracontact connection if and only if it is an Einstein manifold of the form* $S(Y,U) = -(n-1)g(Y,U)$.

Proof. In a para-Kenmotsu manifold M, the Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ and scalar curvature $r_{\nabla^{\mathcal{Z}}}$ of the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ are defined by

$$S_{\nabla^{\mathcal{Z}}}(Y,U) = S(Y,U) + (n-1)g(Y,U), \tag{3.4}$$

$$r_{\nabla^{\mathcal{Z}}} = r + n(n-1), \tag{3.5}$$

where S and r denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively.

Remark 3.2. *For a para-Kenmotsu manifold M with respect to the Zamkovoy canonical paracontact connection* $\nabla^{\mathcal{Z}}$*:*
(a) The curvature tensor $R_{\nabla^{\mathcal{Z}}}$ is given by (3.3),
(b) The Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ is given by (3.4),
(c)$R_{\nabla^{\mathcal{Z}}}(X,\xi)U = R_{\nabla^{\mathcal{Z}}}(\xi,Y)U = R_{\nabla^{\mathcal{Z}}}(X,Y)\xi = 0$,
(d) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) + R'_{\nabla^{\mathcal{Z}}}(X,Y,V,U) = 0$,
(e) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) + R'_{\nabla^{\mathcal{Z}}}(Y,X,V,U) = 0$,
(f) $R'_{\nabla^{\mathcal{Z}}}(X,Y,U,V) - R'_{\nabla^{\mathcal{Z}}}(U,V,X,Y) = 0$,
(g) $R_{\nabla^{\mathcal{Z}}}(X,\xi)U = R_{\nabla^{\mathcal{Z}}}(\xi,Y)U = R_{\nabla^{\mathcal{Z}}}(X,Y)\xi = 0$,
(h) $S_{\nabla^{\mathcal{Z}}}(Y,\xi) = 0$,
(i) The Ricci tensor $S_{\nabla^{\mathcal{Z}}}$ is symmetric,
(j) The scalar curvature $r_{\nabla^{\mathcal{Z}}}$ is given by (3.5).

Next, suppose that a para-Kenmotsu manifold is Ricci flat with respect to the Zamkovoy canonical paracontact connection. Then from (3.4) we get

$$S(Y,U) = -(n-1)g(Y,U). \tag{3.6}$$

Conversely, if the manifold is an Einstein manifold of the form $S(Y,U) = -(n-1)g(Y,U)$, then from (3.4) it follows that $S_{\nabla}(Y,U) = 0$: □

Proposition 3.3. *If in a para-Kenmotsu manifold the curvature tensor of the Zamkovoy canonical paracontact connection vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^{\perp}$ is -1.*

Proof. Let $\xi^{\perp}$ denote the $(n-1)$-dimensional distribution orthogonal to ξ in a para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection whose curvature tensor vanishes. Then for any $X \in \xi^{\perp}$, $g(X,\xi) = 0$ or, $\eta(X) = 0$. Now we shall determine the sectional curvature $'R$ of the plane determine by the vectors $X, Y \in \xi^{\perp}$. Taking inner product on both sides of (3.3) with X and then for $U = Y$, we have

$$R_{\nabla^{\mathcal{Z}}}(X,Y,Y,X) = R(X,Y,Y,X) + g(Y,Y)g(X,X) - g(X,Y)g(X,Y). \tag{3.7}$$

Putting $R_{\nabla^{\mathcal{Z}}} = 0$ in (3.7) we get

$$'R(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = -1.$$

This proves the require result. □

4 Local ϕ-symmetry and local concircular ϕ-symmetry with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇

Definition 4.1. *A para-Kenmotsu manifold is said to be locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its curvature tensor $R_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ satisfies the condition*

$$\phi^2((\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U) = 0, \tag{4.1}$$

for any vector fields X, Y, U, W orthogonal to ξ.

Proposition 4.2. *A para-Kenmotsu manifold is locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Proof. Let us suppose that a para-Kenmotsu manifold M is locally ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. Then, by the help of (3.2), (4.1) simplifies as follow

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U &= (\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U + g(W, R_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi \\ &- \eta(R_{\nabla^{\mathcal{Z}}}(X,Y)U)W + \eta(W)\phi R_{\nabla^{\mathcal{Z}}}(X,Y)U.\end{aligned} \tag{4.2}$$

By virtue of $\eta(R_{\nabla^{\mathcal{Z}}}(X,Y)U)=0$, (4.2) reduces to

$$\begin{aligned}&(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U\\ =\;&(\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U+g(W,R_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi+\eta(W)\phi R_{\nabla^{\mathcal{Z}}}(X,Y)U.\end{aligned}\tag{4.3}$$

Now covariant differentiation of (3.3) with respect to W, we obtain

$$(\nabla_W R_{\nabla^{\mathcal{Z}}})(X,Y)U=(\nabla_W R)(X,Y)U.\tag{4.4}$$

Using (4.4) in (4.3), we get

$$\begin{aligned}&(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U\\ =\;&(\nabla_W R)(X,Y)U+\{R'(X,Y,U,W)+g(Y,U)g(X,W)-g(X,U)g(Y,W)\}\xi\\ +\;&\eta(W)\{\phi R(X,Y)U+g(Y,U)\phi X-g(X,U)\phi Y\}.\end{aligned}\tag{4.5}$$

Applying ϕ^2 on both sides of (4.5); then using (2.1) and (2.2), we obtain

$$\begin{aligned}\phi^2(\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U \;=\;& \phi^2(\nabla_W R)(X,Y)U\\ +\;&\eta(W)\{\phi R(X,Y)U+g(Y,U)\phi X-g(X,U)\phi Y\}.\end{aligned}\tag{4.6}$$

If we consider X,Y,U,W orthogonal to ξ, (4.6) gives to

$$\phi^2((\nabla^{\mathcal{Z}}_W R_{\nabla^{\mathcal{Z}}})(X,Y)U)=\phi^2((\nabla_W R)(X,Y)U).$$

It completes the proof. □

Definition 4.3. *For an n-dimensional $(n>1)$ para-Kenmotsu manifold the concircular curvature tensor $C_{\nabla^{\mathcal{Z}}}$ with respect to the Zamkovoy canonical paracontact connection is defined by*

$$C_{\nabla^{\mathcal{Z}}}(X,Y)U=R_{\nabla^{\mathcal{Z}}}(X,Y)U-\frac{r_{\nabla^{\mathcal{Z}}}}{n(n-1)}[g(Y,U)X-g(X,U)Y].\tag{4.7}$$

where $R_{\nabla^{\mathcal{Z}}}$ and $r_{\nabla^{\mathcal{Z}}}$ are the Riemannian curvature tensor and scalar curvature with respect to the connection $\nabla^{\mathcal{Z}}$, respectively.

Using (3.3) and (3.5) in (4.7), we get

$$C_{\nabla^{\mathcal{Z}}}(X,Y)U = C(X,Y)U,\tag{4.8}$$

where

$$C(X,Y)U=R(X,Y)U-\frac{r}{n(n-1)}[g(Y,U)X-g(X,U)Y]\tag{4.9}$$

is the concircular curvature tensor [24] with respect to the Levi-Civita connection ∇. Thus, the concircular curvature tensor with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇ are equal.

Definition 4.4. *A para-Kenmotsu manifold is said to be locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its concircular curvature tensor $C_{\nabla^{\mathcal{Z}}}$ with respect to the connection $\nabla^{\mathcal{Z}}$ satisfies the condition*

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = 0, \tag{4.10}$$

for any vector fields X, Y, U, W orthogonal to ξ.

Proposition 4.5. *A para-Kenmotsu manifold is locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Proof. If a para-Kenmotsu manifold M is locally concircular ϕ-symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$, then using (3.2), (4.10) simplifies to

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U &= (\nabla_W C_{\nabla^{\mathcal{Z}}})(X,Y)U + g(W, C_{\nabla^{\mathcal{Z}}}(X,Y)U)\xi \\ &- \eta(C_{\nabla^{\mathcal{Z}}}(X,Y)U)W + \eta(W)(\phi C_{\nabla^{\mathcal{Z}}})(X,Y)U. \end{aligned}\tag{4.11}$$

Now covariant differentiation of (4.8) with respect to W, yields

$$(\nabla_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = (\nabla_W C)(X,Y)U. \tag{4.12}$$

Making use of (4.8) and (4.12) in (4.11) we obtain

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = (\nabla_W C)(X,Y)U + g(W, C(X,Y)U)\xi \\ - \eta(C(X,Y)U)W + \eta(W)(\phi C)(X,Y)U. \end{aligned}\tag{4.13}$$

Taking account of (4.9), we write (4.13) as

$$\begin{aligned} &(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U \\ = \; &(\nabla_W C)(X,Y)U + R'(X,Y,U,W)\xi + \eta(W)\phi R(X,Y)U \\ - \; &\frac{r}{n(n-1)}\{g(Y,U)(g(X,W)\xi + \eta(W)\phi X) - g(X,U)(g(Y,W)\xi + \eta(W)\phi Y)\} \\ - \; &\left[\frac{r}{n(n-1)} + 1\right]\{g(X,U)\eta(Y)W - g(Y,U)\eta(X)W\}. \end{aligned}\tag{4.14}$$

Applying ϕ^2 on both sides of above equation; then using (2.1) and (2.2) in (4.14) we have

$$\begin{aligned} &\phi^2(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U \\ = \; &\phi^2(\nabla_W C)(X,Y)U + \eta(W)\phi R(X,Y)U \\ - \; &\frac{r}{n(n-1)}\{g(Y,U)\phi X - g(X,U)\phi Y\}\eta(W) \\ - \; &\left[\frac{r}{n(n-1)} + 1\right]\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\}(W - \eta(W)\xi). \end{aligned}\tag{4.15}$$

If we consider X, Y, U, W orthogonal to ξ, (4.15) reduces to

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = \phi^2((\nabla_W C)(X,Y)U). \tag{4.16}$$

This ends the proof of the required result. □

Proposition 4.6. *Let M be an n-dimensional $(n > 1)$ locally concircular ϕ-symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$. If the scalar curvature r with respect to the Levi-Civita connection ∇ is constant, then M is locally ϕ-symmetric.*

Proof. Now, from (4.9) we have

$$(\nabla_W C)(X,Y)U = (\nabla_W R_\nabla)(X,Y)U - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{4.17}$$

From (4.17) in (4.16) we obtain

$$\phi^2((\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U) = \phi^2((\nabla_W R)(X,Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)\phi^2 X - g(X,U)\phi^2 Y]. \tag{4.18}$$

By virtue of (2.1) in (4.18) and then taking X, Y, U, W orthogonal to ξ, we get

$$\phi^2(\nabla^{\mathcal{Z}}_W C_{\nabla^{\mathcal{Z}}})(X,Y)U = \phi^2((\nabla_W R)(X,Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{4.19}$$

If r is constant, then $\nabla_W r$ is zero. Therefore, (4.19) gives

$$\phi^2(C_{\nabla^{\mathcal{Z}}})(X,Y)U = \phi^2((\nabla_W R)(X,Y)U).$$

Hence, it completes the proof of the required result. □

5 Local ϕ-Ricci symmetry with respect to the connections $\nabla^{\mathcal{Z}}$ and ∇

Definition 5.1. *A para-Kenmotsu manifold M is said to be locally ϕ-Ricci symmetric with respect to the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ if its Ricci operator $Q_{\nabla^{\mathcal{Z}}}$ satisfies*

$$\phi^2((\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X) = 0, \tag{5.1}$$

for any vector fields X, W orthogonal to ξ, and $S_{\nabla^{\mathcal{Z}}}(X,W) = g(Q_{\nabla^{\mathcal{Z}}}X, W)$.

Proposition 5.2. *If a para-Kenmotsu manifold is locally ϕ-Ricci symmetric with respect to the Zamkovoy canonical paracontact connection, then the manifold is Ricci symmetric.*

Proof. Let us consider a para-Kenmotsu manifold, which is locally ϕ-Ricci symmetric with respect to the connection $\nabla^{\mathcal{Z}}$. Then by virtue of (2.1) it follows from (5.1) that

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X - \eta((\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X)\xi = 0. \tag{5.2}$$

From (3.4) we can write

$$Q_{\nabla^{\mathcal{Z}}}X = QX + (n-1)X. \tag{5.3}$$

Again we have

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X = \nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}}X - Q_{\nabla^{\mathcal{Z}}}(\nabla^{\mathcal{Z}}_W X), \tag{5.4}$$

Using (5.3) in (5.4) we get

$$(\nabla^{\mathcal{Z}}_W Q_{\nabla^{\mathcal{Z}}})X = (\nabla^{\mathcal{Z}}_W Q)X. \tag{5.5}$$

Taking account of (5.5), (5.2) reduces to

$$(\nabla^{\mathcal{Z}}_W Q)X - \eta((\nabla^{\mathcal{Z}}_W Q)X)\xi = 0. \tag{5.6}$$

From (3.2) it follows that,

$$\begin{aligned} (\nabla^{\mathcal{Z}}_W Q)X &= \nabla^{\mathcal{Z}}_W QX - Q(\nabla^{\mathcal{Z}}_W X), \\ &= (\nabla_W Q)X + S(W,X)\xi + (n-1)(\eta(X)W + g(W,X)\xi) \\ &+ \eta(X)QW + \eta(W)(\phi QX - Q\phi X) \end{aligned} \tag{5.7}$$

and

$$\eta((\nabla^{\mathcal{Z}}_W Q)X) = \eta((\nabla_W Q)X) + S(W,X) + (n-1)g(W,X). \tag{5.8}$$

Using (5.7) and (5.8) we get from (5.6) that

$$(\nabla_W Q)X + (n-1)\eta(X)W + \eta(W)(\phi QX - Q\phi X) + \eta(X)QW - \eta((\nabla_W Q)X)\xi = 0. \tag{5.9}$$

Taking inner product with U of (5.9) and considering X, W, U orthogonal to ξ, we get

$$(\nabla_W S)(X,U) = 0, \tag{5.10}$$

which implies that the manifold is Ricci symmetric with respect to the Levi-Civita connection ∇. Hence the proof. □

Proposition 5.3. *A locally ϕ-Ricci symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection is an Einstein manifold.*

Proof. Putting $X = \xi$ in (5.10) and using (2.11), we get

$$S(W,U) = -(n-1)g(W,U), \tag{5.11}$$

for any vector fields $W, U \in \chi(M)$.
This ends the required proof. □

6 A para-Kenmotsu manifold M whose curvature tensor is covariant constant with respect to the connection $\nabla^{\mathcal{Z}}$ and M is recurrent with respect to the connection ∇

Definition 6.1. *A para-Kenmotsu manifold M with respect to the Levi-Civita connection is said to be recurrent [11] if its curvature tensor R satisfies the condition.*

$$(\nabla_W R)(X,Y)U = A(W)R(X,Y)U, \tag{6.1}$$

where A is a non-zero 1-form and $X, Y, U, W \in \chi(M)$.

Proposition 6.2. *If in a para-Kenmotsu manifold the curvature tensor is covariant constant with respect to the Zamkovoy canonical paracontact connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an η-Einstein manifold.*

Proof. From (3.2), we can write (6.1) as

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W R)(X,Y)U &= \nabla^{\mathcal{Z}}_W R(X,Y)U - R(\nabla^{\mathcal{Z}}_W X,Y)U - R(X,\nabla^{\mathcal{Z}}_W Y)U - R(X,Y)\nabla^{\mathcal{Z}}_W U,\\ &= (\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi - \eta(R(X,Y)U)W\\ &+ \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\ &+ \eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\ &- g(X,W)R(\xi,Y)U - g(Y,W)R(X,\xi)U - g(U,W)R(X,Y)\xi. \end{aligned}\tag{6.2}$$

Using (2.7)-(2.9) in (6.2), we obtain

$$\begin{aligned}(\nabla^{\mathcal{Z}}_W R)(X,Y)U &= (\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi\\ &+ \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\ &+ \eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\ &- g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\}\\ &- g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(W,U)\{\eta(X)Y - \eta(Y)X\}. \end{aligned}\tag{6.3}$$

Let $(\nabla^{\mathcal{Z}}_W R)(X,Y)U = 0$, then from (6.3), it follows that

$$\begin{aligned}&(\nabla_W R)(X,Y)U + g(W,R(X,Y)U)\xi\\ &+ \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W\\ &+ \eta(W)\{\phi R(X,Y)U - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\}\\ &- g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\}\\ &- g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(W,U)\{\eta(X)Y - \eta(Y)X\} = 0. \end{aligned}\tag{6.4}$$

Now using (6.1) in (6.4), we have

$$\begin{aligned} & A(W)R(X,Y)U + g(W,R(X,Y)U)\xi \\ + \quad & \eta(X)R(W,Y)U + \eta(Y)R(X,W)U + \eta(U)R(X,Y)W \\ + \quad & \eta(W)\{\phi R(X,Y)U) - R(\phi X,Y)U - R(X,\phi Y)U - R(X,Y)\phi U\} \\ - \quad & g(X,U)\eta(Y)W + g(Y,U)\eta(X)W - g(X,W)\{\eta(U)Y - g(Y,U)\xi\} \\ - \quad & g(Y,W)\{g(X,U)\xi - \eta(U)X\} - g(U,W)\{\eta(X)Y - \eta(Y)X\} = 0. \end{aligned} \tag{6.5}$$

Taking the inner product of (6.5) with ξ and using (2.2) and (2.9), it follows that

$$\begin{aligned} & A(W)\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\} + g(W,R(X,Y)U) \\ + \quad & \eta(W)\{g(\phi Y,U)\eta(X) - g(\phi X,U)\eta(Y) - g(X,\phi U)\eta(Y) \\ + \quad & g(Y,\phi U)\eta(X)\} + g(X,W)g(Y,U) - g(Y,W)g(X,U) = 0. \end{aligned} \tag{6.6}$$

Contracrting (6.6) over X and W, we obtain

$$S(Y,U) = \{A(\xi) - (n-1)\}g(Y,U) - A(U)\eta(Y). \tag{6.7}$$

Since the Ricci tensor S with respect to the connection ∇ is symmetric; then from (6.7), we get

$$A(U)\eta(Y) = A(Y)\eta(U). \tag{6.8}$$

Putting $Y = \xi$ in (6.8) and using (2.2) we have

$$A(U) = A(\xi)\eta(U). \tag{6.9}$$

Combining (6.7) and (6.9), it follows that

$$S(Y,U) = \{A(\xi) - (n-1)\}g(Y,U) - A(\xi)\eta(Y)\eta(U). \tag{6.10}$$

This results shows that the manifold is an η-Einstein manifold. Hence the proof. □

7 Example

We consider the 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields (see [27], example of section 7)

$$X = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}, \quad \xi = (x+2y)\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of M^3.

The 1-form $\eta = dz$ defines an almost paracontact structure on M^3 with characteristic vector field ξ. Let g, ϕ be the semi-Riemannian metric and the $(1,1)$ tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -(x+2y) \\ 0 & -1 & (2x+y) \\ -(x+2y) & (2x+y) & 1-(2x+y)^2+(x+2y)^2 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Clearly, (ϕ, ξ, η, g) defines an almost paracontact metric structure on M^3. Let ∇ be the Levi-Civita connection with metric g, then we have

$$[X, \phi X] = 0, \quad [X, \xi] = X + 2\phi X, \quad [\phi X, \xi] = 2X + \phi X.$$

Next, by using the well-known Koszul's formula, we obtain

$$\begin{aligned} &\nabla_X X = -\xi, \quad \nabla_{\phi X} X = 0, \quad \nabla_\xi X = -2\phi X, \\ &\nabla_X \phi X = 0, \quad \nabla_{\phi X} \phi X = \xi, \quad \nabla_\xi \phi X = -2X, \\ &\nabla_X \xi = X, \quad \nabla_{\phi X} \xi = \phi X, \quad \nabla_\xi \xi = 0. \end{aligned}$$

Hence, from the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a para-Kenmotsu manifold. By the above results, one can easily compute

$$\begin{aligned} &R(X, \phi X)\xi = 0, \quad R(\phi X, \xi)\xi = -\phi X, \quad R(X, \xi)\xi = -X, \\ &R(X, \phi X)\phi X = X, \quad R(\phi X, \xi)\phi X = -\xi, \quad R(X, \xi)\phi X = 0, \\ &R(X, \phi X)X = \phi X, \quad R(\phi X, \xi)X = 0, \quad R(X, \xi)X = \xi. \end{aligned} \tag{7.1}$$

Using (7.1), we have constant scalar curvature as follows:

$$r = S(X, X) - S(\phi X, \phi X) + S(\xi, \xi) = -6.$$

Now consider the Zamkovoy canonical paracontact connection $\nabla^{\mathcal{Z}}$ defined by (3.2) such that

$$\begin{aligned} &\nabla^{\mathcal{Z}}_X X = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} X = 0, \quad \nabla^{\mathcal{Z}}_\xi X = -\phi X, \\ &\nabla^{\mathcal{Z}}_X \phi X = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} \phi X = 0, \quad \nabla^{\mathcal{Z}}_\xi \phi X = -X, \\ &\nabla^{\mathcal{Z}}_X \xi = 0, \quad \nabla^{\mathcal{Z}}_{\phi X} \xi = 0, \quad \nabla_\xi \xi = 0. \end{aligned}$$

Again, by the above results we can compute the components of curvature tensors with respect to the connection $\nabla^{\mathcal{Z}}$ as follows:

$$\begin{aligned} &R_{\nabla^{\mathcal{Z}}}(X, \phi X)\xi = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)\xi = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)\xi = 0, \\ &R_{\nabla^{\mathcal{Z}}}(X, \phi X)\phi X = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)\phi X = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)\phi X = 0, \\ &R_{\nabla^{\mathcal{Z}}}(X, \phi X)X = 0, \quad R_{\nabla^{\mathcal{Z}}}(\phi X, \xi)X = 0, \quad R_{\nabla^{\mathcal{Z}}}(X, \xi)X = 0. \end{aligned} \tag{7.2}$$

Using (7.2), we have constant scalar curvature r_{∇^z} as follows:

$$r_{\nabla^z} = S_{\nabla^z}(X, X) - S_{\nabla^z}(\phi X, \phi X) + S_{\nabla^z}(\xi, \xi) = 0.$$

The above arguments easily verifies all the properties of Remark 3.2 and Proposition 3.1.

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